# A note on sublinear separators and expansion 

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#### Abstract

For a hereditary class $\mathcal{G}$ of graphs, let $s_{\mathcal{G}}(n)$ be the minimum function such that each $n$-vertex graph in $\mathcal{G}$ has a balanced separator of order at most $s_{\mathcal{G}}(n)$, and let $\nabla_{\mathcal{G}}(r)$ be the minimum function bounding the expansion of $\mathcal{G}$, in the sense of bounded expansion theory of Nešetřil and Ossona de Mendez. The results of Plotkin, Rao, and Smith (1994) and Esperet and Raymond (2018) imply that if $s_{\mathcal{G}}(n)=\Theta\left(n^{1-\varepsilon}\right)$ for some $\varepsilon>0$, then $\nabla_{\mathcal{G}}(r)=\Omega\left(r^{\frac{1}{2 \varepsilon}-1} /\right.$ polylog $\left.r\right)$ and $\nabla_{\mathcal{G}}(r)=O\left(r^{\frac{1}{\varepsilon}-1}\right.$ polylog $\left.r\right)$. Answering a question of Esperet and Raymond, we show that neither of the exponents can be substantially improved.


For an $n$-vertex graph $G$, a set $X \subseteq V(G)$ is a balanced separator if each component of $G-X$ has at most $2 n / 3$ vertices. Let $s(G)$ denote the minimum size of a balanced separator in $G$, and for a class $\mathcal{G}$ of graphs, let $s_{\mathcal{G}}: \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$
s_{\mathcal{G}}(n)=\max \{s(G): G \in \mathcal{G},|V(G)| \leq n\} .
$$

Let us remark that this notion is related to the separation profile studied for infinite graphs [1]. Classes with sublinear separators (i.e., classes $\mathcal{G}$ with $\left.s_{\mathcal{G}}(n)=o(n)\right)$ are of interest from the computational perspective, as they naturally admit divide-and-conquer style algorithms. They also turn out to have a number of intriguing structural properties; the relevant one for this note is the connection to the density of shallow minors.

For a graph $G$ and an integer $r \geq 0$, an $r$-shallow minor of $G$ is any graph obtained from a subgraph of $G$ by contracting pairwise vertex-disjoint subgraphs, each of radius at most $r$. The density of a graph $H$ is $|E(H)| /|V(H)|$.

[^0]We let $\nabla_{r}(G)$ denote the maximum density of an $r$-shallow minor of $G$. For a class $\mathcal{G}$ of graphs, let $\nabla_{\mathcal{G}}: \mathbb{N} \rightarrow \mathbb{R} \cup\{\infty\}$ be defined by

$$
\nabla_{\mathcal{G}}(r)=\sup \left\{\nabla_{r}(G): G \in \mathcal{G}\right\} .
$$

If $\nabla_{\mathcal{G}}(r)$ is finite for every $r$, we say that the class $\mathcal{G}$ has bounded expansion. The classes with bounded expansion have a number of common properties and computational applications; we refer the reader to [10] for more details. The first connection between sublinear separators and bounded expansion comes from the work of Plotkin, Rao, and Smith [11.

Theorem 1 (Plotkin, Rao, and Smith [11]). For each n-vertex graph $G$ $(n \geq 2)$ and all integers $l, h \geq 1$, either $G$ has a balanced separator of order at most $n / l+2 h^{2} l \log _{2} n$, or $G$ contains a $\left(2 l \log _{2} n\right)$-shallow minor of $K_{h}$.

As observed in [6] (and qualitatively in [9, 4]), this has the following consequence.

Corollary 2. Suppose $\mathcal{G}$ is a class of graphs such that $s_{\mathcal{G}}(n)=\Omega\left(n^{1-\varepsilon}\right)$ for some $\varepsilon>0$. Then $\nabla_{\mathcal{G}}(r)=\Omega\left(r^{\frac{1}{2 \varepsilon}-1} /\right.$ polylog $\left.r\right)$.

Proof. Consider a sufficiently large integer $r$, and let $n=\left\lfloor r^{\frac{1}{\varepsilon}} \log _{2}^{-2 / \varepsilon} r\right\rfloor$, $l=\left\lfloor\frac{1}{2} r \log _{2}^{-1} n\right\rfloor$, and $h=\left\lfloor\frac{1}{2} n^{1 / 2} l^{-1} \log _{2}^{-1} n\right\rfloor$. Note that

$$
\begin{aligned}
2 h^{2} l \log _{2} n & <n / l \\
n^{\varepsilon} & \leq r \log _{2}^{-2} r \leq r \\
\log _{2} n & \leq \frac{1}{\varepsilon} \log _{2} r \\
2 l \log _{2} n & \leq r \leq 3 l \log _{2} n .
\end{aligned}
$$

Consequently, we have

$$
\frac{n}{l}+2 h^{2} l \log _{2} n \leq \frac{2 n}{l} \leq \frac{6 n \log _{2} n}{r} \leq 6 n^{1-\varepsilon} \frac{\log _{2} n}{\log _{2}^{2} r} \leq \frac{6}{\varepsilon \log _{2} r} n^{1-\varepsilon}<s_{\mathcal{G}}(n) .
$$

Let $G \in \mathcal{G}$ be a graph with at most $n$ vertices and with no balanced separator of order less than $s_{\mathcal{G}}(n)$; then Theorem 1 implies $G$ contains an $r$-shallow minor of $K_{h}$, implying that

$$
\nabla_{\mathcal{G}}(r) \geq \nabla_{r}(G) \geq \frac{\left|E\left(K_{h}\right)\right|}{\left|V\left(K_{h}\right)\right|}=\Omega(h)=\Omega\left(r^{\frac{1}{2 \varepsilon}-1} / \text { polylog } r\right) .
$$

Dvořák and Norin 4 proved that surprisingly, a converse to Corollary 2 holds as well. Subsequently, Esperet and Raymond [6] gave a simpler argument with a better exponent: they state their result with $O\left(r^{\frac{1}{\varepsilon}}\right.$ polylog $r$ ) bound, but an analysis of their argument shows that the exponent can be improved by 1. We include the short proof for completeness; the proof uses the following result establishing the connection between separators and treewidth.

Theorem 3 (Dvořák and Norin [5]). Let $G$ be a graph and $k$ an integer. If $s(H) \leq k$ for every induced subgraph $H$ of $G$, then $t w(G) \leq 15 k$.

We also need a simple observation on shallow minors and subdivisions.
Observation 4. For every integer $r \geq 0$, if $H$ is an $r$-shallow minor of $a$ graph $G$ and $H$ has maximum degree at most three, then a subgraph of $G$ is isomorphic to a graph obtained from $H$ by subdividing each edge at most $4 r$ times.

Proof. Since $H$ has maximum degree at most three, we can without loss of generality assume each subgraph $S_{v}$ contracted to form a vertex $v \in V(H)$ is a subdivision of a star with at most three rays. Since $S_{v}$ has radius at most $r$, the distance from the center of $S_{v}$ to each leaf is at most $2 r$. For each edge $u v \in E(H)$, let $e_{u v}$ be an edge of $G$ with one end in $S_{u}$ and the other end in $S_{v}$; then $S_{u}+e_{u v}+S_{v}$ contains a path from the center of $S_{u}$ to the center of $S_{v}$ of length at most $4 r+1$. Consequently, the union of the graphs $S_{v}$ for $v \in V(H)$ and the edges $e_{u v}$ for $u v \in E(H)$ gives a subgraph of $G$ is isomorphic to a graph obtained from $H$ by subdividing each edge at most $4 r$ times.

For $\alpha>0$, a graph $G$ is an $\alpha$-expander if $|N(S)| \geq \alpha|S|$ holds for every set $S \subseteq V(G)$ of size at most $|V(G)| / 2$.

Theorem 5 (Esperet and Raymond [6]). Suppose $\mathcal{G}$ is a hereditary class of graphs such that $s_{\mathcal{G}}(n)=O\left(n^{1-\varepsilon}\right)$ for some $\varepsilon>0$. Then $\nabla_{\mathcal{G}}(r)=$ $O\left(r^{\frac{1}{\varepsilon}-1}\right.$ polylog $\left.r\right)$.

Proof. Let $H$ be an $r$-shallow minor of a graph $G \in \mathcal{G}$ and let $d$ be the density of $H$. By the result of Shapira and Sudakov [13], there exists a subgraph $H_{1} \subseteq H$ of average degree $\Omega(d)$ such that, letting $n=\left|V\left(H_{1}\right)\right|$, the graph $H_{1}$ is a $(1 / \operatorname{polylog} n)$-expander. Consequently, $H_{1}$ has treewidth $\Omega(n / \operatorname{polylog} n)$. As Chekuri and Chuzhoy [3] proved, $H_{1}$ has a subcubic subgraph $H_{2}$ of treewidth $\Omega(n / \operatorname{poly} \log n)$. Since $H_{2}$ is a subcubic
$r$-shallow minor of $G$, Observation 4 implies that $G$ has a subgraph $G_{2}$ obtained from $H_{2}$ by subdividing each edge at most $4 r$ times, and thus $\left|V\left(G_{2}\right)\right|=O\left(r\left|V\left(H_{2}\right)\right|\right)=O(r n)$. Furthermore, since $H_{2}$ is a minor of $G_{2}$, we have

$$
\begin{equation*}
\operatorname{tw}\left(G_{2}\right) \geq \operatorname{tw}\left(H_{2}\right)=\Omega(n / \text { polylog } n) . \tag{1}
\end{equation*}
$$

On the other hand, since $\mathcal{G}$ is hereditary, $G_{2}$ is a spanning subgraph of a graph from $\mathcal{G}$, and thus every subgraph of $G_{2}$ has a balanced separator of size $O\left(\left|V\left(G_{2}\right)\right|^{1-\varepsilon}\right)=O\left((r n)^{1-\varepsilon}\right)$. By Theorem 3 this implies $\operatorname{tw}\left(G_{2}\right)=O\left((r n)^{1-\varepsilon}\right)$. Combining this inequality with (1), this gives $n=$ $O\left(r^{\frac{1}{\varepsilon}-1} \operatorname{poly} \log n\right)=O\left(r^{\frac{1}{\varepsilon}-1}\right.$ polylog $\left.r\right)$. Since $H_{1}$ has $n$ vertices and average degree $\Omega(d)$, we have $d=O(n)=O\left(r^{\frac{1}{\varepsilon}-1}\right.$ polylog $\left.r\right)$. This holds for every $r$-shallow minor of a graph from $\mathcal{G}$, and thus $\nabla_{\mathcal{G}}(r)=O\left(r^{\frac{1}{\varepsilon}-1} \operatorname{polylog} r\right)$.

For $0<\varepsilon \leq 1$,

- let $b_{\varepsilon}$ denote the supremum of real numbers $b$ for which every hereditary class $\mathcal{G}$ of graphs such that $s_{\mathcal{G}}(n)=\Theta\left(n^{1-\varepsilon}\right)$ satisfies $\nabla_{\mathcal{G}}(r)=$ $\Omega\left(r^{b}\right)$, and
- let $B_{\varepsilon}$ denote the infimum of real numbers $B$ for which every hereditary class $\mathcal{G}$ of graphs such that $s_{\mathcal{G}}(n)=\Theta\left(n^{1-\varepsilon}\right)$ satisfies $\nabla_{\mathcal{G}}(r)=O\left(r^{B}\right)$.

Corollary 2 and Theorem 5 give the following bounds.
Corollary 6. For $0<\varepsilon \leq 1$,

$$
\max \left(\frac{1}{2 \varepsilon}-1,0\right) \leq b_{\varepsilon} \leq B_{\varepsilon} \leq \frac{1}{\varepsilon}-1
$$

Esperet and Raymond [6] asked whether either of these bounds (in particular, in terms of multiplicative constants) can be improved. They suggest some insight into this question could be obtained by investigating the $d$ dimensional grids. While the grids ultimately do not give the best bounds we obtain, their analysis is instructive and we give it (for even $d$ ) in the following lemma.

Note that $b_{1 / 2}=0$, matching the lower bound from Corollary 6. Indeed, as proved by Lipton and Tarjan [7, the class $\mathcal{P}$ of planar graphs satisfies $s_{\mathcal{P}}(n)=\Theta\left(n^{1 / 2}\right)$, and on the other hand, every minor of a planar graph is planar, implying $\nabla_{\mathcal{P}}(r) \leq 3=O\left(r^{0}\right)$. However, 2-dimensional grids with diagonals give $B_{1 / 2}=1$, as we will show in greater generality in the next lemma. Hence, $b_{\varepsilon}$ is not always equal to $B_{\varepsilon}$.

Lemma 7. For every even integer d,

$$
b_{1 / d} \leq \frac{d}{2} \leq B_{1 / d}
$$

Proof. Let $Q_{n}^{d}$ denote the graph whose vertices are elements of $\{1, \ldots, n\}^{d}$ and two distinct vertices are adjacent if they differ by at most 2 in each coordinate. Let $\mathcal{G}_{d}$ denote the class consisting of graphs $Q_{n}^{d}$ for all $n \in \mathbb{N}$ and their induced subgraphs. Note that $s_{\mathcal{G}}(n)=\Theta\left(n^{1-1 / d}\right)$ : Each induced subgraph $H$ of $Q_{n}^{d}$ can be represented as an intersection graph of axis-aligned unit cubes in $\mathbb{R}^{d}$ where each point is contained in at most $3^{d}$ cubes, and such graphs have balanced separators of order $O\left(|V(H)|^{1-1 / d}\right)$, see e.g. 8]. Conversely, standard isoperimetric inequalities show that $Q_{n}^{d}$ does not have a balanced separator smaller than $\Omega\left(n^{d-1}\right)=\Omega\left(\left|V\left(Q_{n}^{d}\right)\right|^{1-1 / d}\right)$. We claim that $\nabla_{\mathcal{G}}(r)=\Theta\left(r^{d / 2}\right)$.

Consider any $r$-shallow minor $H$ of $Q_{n}^{d}$, and for $v \in V(H)$, let $B_{v}$ denote the subgraph of $Q_{n}^{d}$ of radius at most $r$ contracted to form $v$. We have $\Delta\left(Q_{n}^{d}\right)<5^{d}$, and thus $\operatorname{deg}_{H}(v)<5^{d}\left|V\left(B_{v}\right)\right|$ holds for every $v \in V(H)$. Let $v$ be the vertex of $H$ with $\left|V\left(B_{v}\right)\right|$ minimum, and let $c$ be a vertex of $Q_{n}^{d}$ such that each vertex of $B_{v}$ is at distance at most $r$ from $c$. Note that if $u v \in V(H)$, then every vertex of $B_{u}$ is at distance at most $3 r+1$ from $c$. Consequently, the pairwise vertex-disjoint subgraphs $B_{u}$ for $u \in N(v)$ are all contained in a cube with side of length $12 r+4$ centered at $c$, implying

$$
\left(\operatorname{deg}_{H}(v)+1\right)\left|V\left(B_{v}\right)\right| \leq\left|V\left(B_{v}\right)\right|+\sum_{u \in N(v)}\left|V\left(B_{u}\right)\right| \leq(12 r+5)^{d},
$$

and thus $\operatorname{deg}_{H}(v)<(12 r+5)^{d} /\left|V\left(B_{v}\right)\right|$. Therefore, since $\min (a x, b / x) \leq$ $\sqrt{a b}$ for every $a, b, x>0$, we have

$$
\operatorname{deg}_{H}(v)<\min \left(5^{d}\left|V\left(B_{v}\right)\right|,(12 r+5)^{d} /\left|V\left(B_{v}\right)\right|\right) \leq(60 r+25)^{d / 2}
$$

Hence, each $r$-shallow minor of $Q_{n}^{d}$ has minimum degree $O\left(r^{d / 2}\right)$, and thus we have $\nabla_{\mathcal{G}}(r)=O\left(r^{d / 2}\right)$.

On the other hand, consider the graph $Q_{2 r}^{d}$. For $x \in\{1, \ldots, 2 r\}^{d / 2}$, let $A_{x}$ be the subgraph of $Q_{2 r}^{d}$ induced by vertices $\left(i_{1}, \ldots, i_{d}\right)$ such that $i_{j}=x_{j}$ for $j=1, \ldots, d / 2$ and $i_{j} \in\{1, \ldots, 2 r\}$ for $j=d / 2+1, \ldots, d$, and let $B_{x}$ be the subgraph induced by vertices $\left(i_{1}, \ldots, i_{d}\right)$ such that $i_{1} \in\{2,4, \ldots, 2 r\}, i_{j} \in$ $\{1, \ldots, 2 r\}$ for $j=2, \ldots, d / 2$, and $i_{j}=x_{j-d / 2}$ for $j=d / 2+1, \ldots, d$. Each of these subgraphs has radius at most $r$, for all distinct $x, x^{\prime} \in\{1, \ldots, 2 r\}^{d / 2}$ we have $V\left(A_{x}\right) \cap V\left(A_{x^{\prime}}\right)=\emptyset$ and $V\left(B_{x}\right) \cap V\left(B_{x^{\prime}}\right)=\emptyset$, and for all $x \in$
$\{1, \ldots, 2 r\}^{d / 2}$ such that $x_{1}$ is odd and $y \in\{1, \ldots, 2 r\}^{d / 2}$, the graphs $A_{x}$ and $B_{y}$ are vertex-disjoint and $Q_{2 r}^{d}$ contains an edge with one end $(x, y) \in V\left(A_{x}\right)$ and the other end in $\left(x+e_{1}, y\right) \in V\left(B_{y}\right)$. Consequently, $K_{(2 r)^{d / 2} / 2,(2 r)^{d / 2}}$ is an $r$-shallow minor of $Q_{2 r}^{d}$, implying $\nabla_{\mathcal{G}}(r)=\Omega\left(r^{d / 2}\right)$.

Lemma 7 implies that $b_{\varepsilon} \leq \frac{1}{2 \varepsilon}$ when $\frac{1}{\varepsilon}$ is an even integer, and thus at these points the lower bound from Corollary 6 cannot be improved by more than 1. Actually, we can prove an even better bound for all values of $\varepsilon>0$. To this end, let us first establish bounds on the size of balanced separators in certain graph classes.

Lemma 8. Let $f, t: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$be non-decreasing functions, and let $p:$ $\mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$be the inverse to the function $x \mapsto x t(x)$. Let $G$ be a graph such that every induced subgraph $H$ of $G$ satisfies $s(H) \leq f(|V(H)|)$. Let $G^{\prime}$ be a graph obtained from $G$ by subdividing each edge at least $t(|V(G)|)$ times. Then $s\left(H^{\prime}\right) \leq 15 f\left(p\left(2\left|V\left(H^{\prime}\right)\right|\right)\right)+1$ for every induced subgraph $H^{\prime}$ of $G^{\prime}$.

Proof. Without loss of generality, we can assume $H^{\prime}$ is connected, as otherwise it suffices to consider the size of a balanced separator in the largest component of $H^{\prime}$. Let $B$ be the set of vertices of $G^{\prime}$ created by subdividing the edges, and let $A=V\left(H^{\prime}\right) \backslash B$ and $a=|A|$. If $a \leq 1$, then $H^{\prime}$ is a tree, and thus it has balanced separator of size at most 1. Hence, assume that $a \geq 2$. Since $H^{\prime}$ is connected, we have $\left|V\left(H^{\prime}\right)\right| \geq(a-1) t(|V(G)|) \geq a t(a) / 2$, and thus $a \leq p\left(2\left|V\left(H^{\prime}\right)\right|\right)$. Note that $H^{\prime}$ is obtained from a subgraph $H$ of $G$ with $a$ vertices by subdividing edges and repeatedly adding pendant vertices. By Theorem 3 we have $\operatorname{tw}(H) \leq 15 f(a)$, and thus $\operatorname{tw}\left(H^{\prime}\right) \leq \operatorname{tw}(H) \leq$ $15 f(a) \leq 15 f\left(p\left(2\left|V\left(H^{\prime}\right)\right|\right)\right)$. As proved in [12], every graph of treewidth at most $c$ has a balanced separator of order at most $c+1$. Consequently, $H^{\prime}$ has a balanced separator of order at most $15 f\left(p\left(2\left|V\left(H^{\prime}\right)\right|\right)\right)+1$.

For a graph $G$ with $m$ vertices and $0<\varepsilon<1$, let $G^{\varepsilon}$ denote the graph obtained from $G$ by subdividing each edge $\left\lceil m^{\varepsilon /(1-\varepsilon)}\right\rceil$ times. For a class of graphs $\mathcal{G}$, let $\mathcal{G}^{\varepsilon}$ denote the class consisting of all induced subgraphs of the graphs $G^{\varepsilon}$ for $G \in \mathcal{G}$.

Lemma 9. For every class of graphs $\mathcal{G}$ and every $0<\varepsilon<1$, we have $s_{\mathcal{G}^{\varepsilon}}(n)=O\left(n^{1-\varepsilon}\right)$. If $\mathcal{G}$ contains all 3 -regular graphs, then $s_{\mathcal{G}^{\varepsilon}}(n)=\Omega\left(n^{1-\varepsilon}\right)$.

Proof. Applying Lemma 8 with $f(n)=n$ and $t(m)=\left\lceil m^{\varepsilon /(1-\varepsilon)}\right\rceil$ (so that $\left.p(n)=\Theta\left(n^{1-\varepsilon}\right)\right)$, we have $s_{\mathcal{G}^{\varepsilon}}(n)=O\left(n^{1-\varepsilon}\right)$.

Conversely, let $G \in \mathcal{G}$ be a 3-regular $\frac{3}{20}$-expander with $m=\Theta\left(n^{1-\varepsilon}\right)$ vertices (such a graph exists for every sufficiently large even number of
vertices [2]). Note that $\left|V\left(G^{\varepsilon}\right)\right|=\Theta\left(m \cdot m^{\varepsilon /(1-\varepsilon)}\right)=\Theta\left(m^{1 /(1-\varepsilon)}\right)=\Theta(n)$. We now argue that $s\left(G^{\varepsilon}\right)=\Omega(m)$, which implies $s_{\mathcal{G}}(n)=\Omega(m)=\Omega\left(n^{1-\varepsilon}\right)$.

Let $M$ be the set of vertices of $G^{\varepsilon}$ of degree three. Suppose for a contradiction $X$ is a balanced separator in $G^{\varepsilon}$ of size $o(m)$. For sufficiently large $n$, this implies $V\left(G^{\varepsilon}\right)$ can be expressed as disjoint union of $X, C_{1}$, and $C_{2}$, where $C_{1}$ and $C_{2}$ are unions of components of $G^{\varepsilon}-X$ and $\left|C_{1}\right|,\left|C_{2}\right| \geq\left|V\left(G^{\varepsilon}\right)\right| / 4=\Omega(n)$. Each component of $G^{\varepsilon}-X$ disjoint from $M$ has two neighbors in $X$, implying the total number of vertices in such components is at most $\frac{3}{2}|X|\left\lceil m^{\varepsilon /(1-\varepsilon)}\right\rceil=o(n)$. Furthermore, a component of $G-X$ containing $k \geq 1$ vertices of $M$ has $O\left(k m^{\varepsilon /(1-\varepsilon)}\right)$ vertices. Consequently, $\left|C_{1} \cap M\right|,\left|C_{2} \cap M\right|=\Omega\left(n / m^{\varepsilon /(1-\varepsilon)}\right)=\Omega(m)$. By symmetry, we can assume $\left|C_{1} \cap M\right| \leq m / 2$, and since $G$ is a $\frac{3}{20}$-expander, we have $N_{G}\left(C_{1} \cap M\right)=\Omega(m)$. However, this implies $|X|=\Omega(m)$, which is a contradiction.

Applying this lemma with $\mathcal{G}$ consisting of all 3-regular graphs, we obtain the following bound.
Lemma 10. For $0<\varepsilon \leq 1, b_{\varepsilon} \leq \frac{1}{2 \varepsilon}-\frac{1}{2}$.
Proof. We have $b_{1}=0$ by Corollary 6, and thus we can assume $\varepsilon<1$. Let $\mathcal{G}_{3}$ be the class of all 3-regular graphs. By Lemma 9, we have $s_{\mathcal{G}_{3}^{\varepsilon}}(n)=\Theta\left(n^{1-\varepsilon}\right)$.

Let $G$ be a 3 -regular graph with $m$ vertices, and consider any $r$-shallow minor $F$ of $G^{\varepsilon}$. If $4 r<\left\lceil m^{\varepsilon /(1-\varepsilon)}\right\rceil$, then $F$ is 2-degenerate, and thus it has density at most 2 . Hence, we can assume $r=\Omega\left(m^{\varepsilon /(1-\varepsilon)}\right)$. Let $M$ be the set of vertices of $G^{\varepsilon}$ of degree three, for each vertex $v \in V(F)$ let $B_{v}$ be the vertex set of the subgraph of $G^{\varepsilon}$ contracted to $v$, and let $v$ be the vertex of $F$ with $\left|B_{v} \cap M\right|$ minimum. Note that $\operatorname{deg}_{F}(v) \leq 2+\left|B_{v} \cap M\right|$. Furthermore, since the sets $B_{u}$ for $u \in V(F)$ are pairwise disjoint, we have $\left(\operatorname{deg}_{F}(v)+\right.$ 1) $\left|B_{v} \cap M\right| \leq\left|B_{v} \cap M\right|+\sum_{u \in N(v)}\left|B_{u} \cap M\right| \leq|M|=m$. Consequently, $\operatorname{deg}_{F}(v) \leq 2+\min \left(\left|B_{v} \cap M\right|, m /\left|B_{v} \cap M\right|\right) \leq 2+\sqrt{m}=O\left(r^{\frac{1}{2 \varepsilon}-\frac{1}{2}}\right)$. Therefore, $\nabla_{\mathcal{G}_{3}^{\varepsilon}}(r)=O\left(r^{\frac{1}{2 \varepsilon}-\frac{1}{2}}\right)$.

Furthermore, note that if $G$ is an expander, then $G$ contains as an $O(\log m)$-shallow minor a clique with $\Omega(\sqrt{m / \log m})$ vertices by Theorem 1 , and thus if $m=\Theta\left(r^{\frac{1}{\varepsilon}-1}\right)$, we conclude $G^{\varepsilon}$ contains as an $O(r \log r)$-shallow minor a clique with $\Omega\left(r^{\frac{1}{2 \varepsilon}}-\frac{1}{2} /\right.$ polylog $\left.r\right)$ vertices. Consequently, $\nabla_{\mathcal{G}_{3}^{\varepsilon}}(r)=$ $\Omega\left(r^{\frac{1}{2 \varepsilon}-\frac{1}{2}} / \operatorname{polylog} r\right)$; hence, the analysis of this example cannot be substantially improved.

This construction also gives a lower bound for $B_{\varepsilon}$ that matches the upper bound from Corollary 6

Lemma 11. For $0<\varepsilon \leq 1, B_{\varepsilon} \geq \frac{1}{\varepsilon}-1$.
Proof. Since $B_{1}=0$ by Corollary6, we can assume $\varepsilon<1$. Let $\mathcal{G}_{a}$ be the class of all graphs. By Lemma 9, we have $s_{\mathcal{G}_{a}^{\varepsilon}}(n)=\Theta\left(n^{1-\varepsilon}\right)$. For a sufficiently large integer $r$, let $m=\left\lfloor r^{\frac{1}{\varepsilon}-1}\right\rfloor$. The graph $K_{m}^{\varepsilon}$ contains the clique $K_{m}$ as an $r$-shallow minor, implying $\nabla_{\mathcal{G}_{a}^{\varepsilon}}(r)=\Omega\left(r^{\frac{1}{\varepsilon}-1}\right)$.

Finally, a similar idea enables us to obtain a better bound for $b_{\varepsilon}$ in the range $\frac{1}{2} \leq \varepsilon \leq 1$.

Lemma 12. For $\frac{1}{2} \leq \varepsilon \leq 1, b_{\varepsilon}=0$.
Proof. Since $b_{1}=0$ by Corollary 6, we can assume $\varepsilon<1$. For a graph $G$ with $m$ vertices, let $G^{\prime}$ denote the graph obtained from $G$ by subdividing each edge $\left\lceil m^{\frac{2 \varepsilon-1}{2-2 \varepsilon}}\right\rceil$ times. Let $\mathcal{G}$ consist of all induced subgraphs of the graphs $G^{\prime}$ for all planar graphs $G$. All graphs in $\mathcal{G}$ are planar, and thus $\nabla_{\mathcal{G}}(r) \leq 3=O\left(r^{0}\right)$ holds for every $r \geq 0$. Standard isoperimetric inequalities applied with $G$ being a $(t \times t)$-grid for $t=\Theta\left(n^{1-\varepsilon}\right)$ (so that $\left|V\left(G^{\prime}\right)\right|=\Theta(n)$ ) show that every balanced separator in $G^{\prime}$ has size $\Omega(t)=\Omega\left(n^{1-\varepsilon}\right)$, implying $s_{\mathcal{G}^{\prime}}(n)=$ $\Omega\left(n^{1-\varepsilon}\right)$. Conversely, Lemma 8 applied with $f(n)=O(\sqrt{n})$ and $t(m)=$ $\left\lceil m^{\frac{2 \varepsilon-1}{2-2 \varepsilon}}\right\rceil$ (so that $p(n)=\Theta\left(n^{2-2 \varepsilon}\right)$ ) implies $s_{\mathcal{G}^{\prime}}(n)=O\left(n^{1-\varepsilon}\right)$.

Let us summarize our findings: We have $b_{\varepsilon}=0$ when $\frac{1}{2} \leq \varepsilon \leq 1$,

$$
\frac{1}{2 \varepsilon}-1 \leq b_{\varepsilon} \leq \frac{1}{2 \varepsilon}-\frac{1}{2}
$$

when $0<\varepsilon<\frac{1}{2}$, and

$$
B_{\varepsilon}=\frac{1}{\varepsilon}-1
$$

when $0<\varepsilon \leq 1$. In particular, if $\varepsilon<1$, then $b_{\varepsilon} \neq B_{\varepsilon}$.
The bounds for $b_{\varepsilon}$ differ by at most $1 / 2$. It is unclear whether the upper or the lower bound can be improved. While the fact that $b_{1 / 2}=0$ matches the lower bound suggests that a better construction improving the upper bound in general could exist, it is also plausible that this is just a "dimension 2" artifact and in fact the lower bound might be possible to improve for $\varepsilon<1 / 2$ (possibly leading to discontinuity of $b_{\varepsilon}$ at $\varepsilon=1 / 2$ ).

Instead of $b_{\varepsilon}$ and $B_{\varepsilon}$, the following parameters (constraining $s_{\mathcal{G}}(n)$ from below by $\Omega\left(n^{1-\varepsilon}\right)$ and from above by $O\left(n^{1-\varepsilon}\right)$, rather than by $\Theta\left(n^{1-\varepsilon}\right)$ ) might be considered more natural. For $0<\varepsilon \leq 1$,

- let $b_{\varepsilon}^{\prime}$ denote the supremum of real numbers $b^{\prime}$ for which every hereditary class $\mathcal{G}$ of graphs such that $s_{\mathcal{G}}(n)=\Omega\left(n^{1-\varepsilon}\right)$ satisfies $\nabla_{\mathcal{G}}(r)=$ $\Omega\left(r^{b^{\prime}}\right)$, and
- let $B_{\varepsilon}^{\prime}$ denote the infimum of real numbers $B^{\prime}$ for which every hereditary class $\mathcal{G}$ of graphs such that $s_{\mathcal{G}}(n)=O\left(n^{1-\varepsilon}\right)$ satisfies $\nabla_{\mathcal{G}}(r)=$ $O\left(r^{B^{\prime}}\right)$.

By Corollary 2 and Theorem 5 5 , we have

$$
\max \left(\frac{1}{2 \varepsilon}-1,0\right) \leq b_{\varepsilon}^{\prime} \leq b_{\varepsilon} \leq B_{\varepsilon} \leq B_{\varepsilon}^{\prime} \leq \frac{1}{\varepsilon}-1
$$

Hence, $b_{\varepsilon}^{\prime}=b_{\varepsilon}=0$ when $\frac{1}{2} \leq \varepsilon \leq 1$,

$$
\frac{1}{2 \varepsilon}-1 \leq b_{\varepsilon}^{\prime} \leq b_{\varepsilon} \leq \frac{1}{2 \varepsilon}-\frac{1}{2}
$$

when $0<\varepsilon<\frac{1}{2}$, and

$$
B_{\varepsilon}^{\prime}=B_{\varepsilon}=\frac{1}{\varepsilon}-1
$$

when $0<\varepsilon \leq 1$. It seems likely that $b_{\varepsilon}^{\prime}=b_{\varepsilon}$ when $0<\varepsilon<\frac{1}{2}$ as well, but this is not obvious: Consider any hereditary class $\mathcal{G}^{\prime}$ such that $s_{\mathcal{G}^{\prime}}(n)=$ $\Omega\left(n^{1-\varepsilon}\right)$. Without loss of generality, we can assume $\mathcal{G}^{\prime}$ is monotone (closed under subgraphs) rather than just hereditary. To try to transform $\mathcal{G}^{\prime}$ to a hereditary class $\mathcal{G}$ with $s_{\mathcal{G}}(n)=\Theta\left(n^{1-\varepsilon}\right)$, it is natural to let $\mathcal{G}$ consist of all graphs $G \in \mathcal{G}$ such that every induced subgraph $H$ of $G$ satisfies $s(H)=O\left(|V(H)|^{1-\varepsilon}\right)$. However, as we have to ensure this bound holds for all induced subgraphs $H$ (in order for $\mathcal{G}$ to be hereditary), it is not clear the remaining graphs are sufficient to enforce $s_{\mathcal{G}}(n)=\Omega\left(n^{1-\varepsilon}\right)$.

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